Indian Statistical Institute, Bangalore
B. Math (Hons.) Third Year
First Semester - Probability III
Mid-Semester Exam

Total Marks: 30 **Date:** September 10, 2026
Maximum Marks: 25 **Duration:** 2 hours

Justify all your answers. *Good luck!*

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1. Provide justifications for whether the following statements are **true or false**.

   (a) Suppose $(\Omega, \mathcal{F})$ is a measurable space. Let $X : \Omega \to \mathbb{R}$ be such that for every rational $q \in \mathbb{Q}$, we have
   $$\{w \in \Omega : X(w) < q\} \in \mathcal{F}.$$
   Then X is measurable. [4]

   (b) Let $X, Y, Z : (\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}$ be random variables. Suppose Y and Z have the same distribution, i.e., $Y \stackrel{d}{=} Z$. Then $X + Y \stackrel{d}{=} X + Z$. [4]

   (c) Let μ be any Borel probability measure on $\mathbb{R}^2$. Then for every $\epsilon > 0$, there exists a compact set $K_\epsilon \subset \mathbb{R}^2$ such that $\mathbb{P}(K_\epsilon) \geq 1 - \epsilon$. [4]

2. Let $(X_n)_{n \in \mathbb{N}}$ be a sequence of random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that
   $$\mathbb{P}(X_n = x) = \begin{cases} \frac{1}{2^n}, & \text{if } x = 1, \\ \frac{1}{4^n}, & \text{if } x = 2, \\ 1 - \frac{1}{4^n} - \frac{1}{2^n}, & \text{if } x = 3. \end{cases}$$
   Let μ_n be the distribution of X_n, i.e., $\mu_n = \mathbb{P} \circ X_n^{-1}$.

   (a) Find measurable $T_n : ([0,1], \mathcal{B}[0,1], \lambda) \to \mathbb{R}$ such that $T_n \stackrel{d}{=} X_n$. Here λ is the Lebesgue measure on $[0,1]$. [3]

   (b) Prove that there is a probability measure μ such that $\mu_n \xrightarrow{d} \mu$, and find μ. [5]

   (c) Let $A_n = \{X_n = 3\}$. Find $\mathbb{P}\left(\limsup_{n \to \infty} A_n\right)$. [5]

3. Let $X : (\Omega, \mathcal{F}, \mathbb{P}) \to [0, \infty)$ be a non-negative random variable. Prove that [5]
   $$\mathbb{E}[X] < \infty \text{ if and only if } \sum_{k=1}^{\infty} \mathbb{P}(X \geq k) < \infty.$$